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ON THE QUANTUM L-OPERATOR FOR THE TWO-DIMENSIONAL LATTICE TODA MODEL

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We consider the two-dimensional quantum lattice Toda model for affine and simple Lie algebras of type A. For its known L-operator, the second-order correction in lattice parameter ε is found. It is proved that the equation determining the third-order correction in ε has no solutions. Bibliography: 9 titles.

1. Introduction

1.1. Continuous classical model. The (1+1)-dimensional Toda chain associated with the affine Lie algebra $A^{(1)}_{N-1}$ is a model which describes the relativistic dynamics of N scalar fields, ϕ_a, $a = 1, \ldots, N$, assigned to nodes of the corresponding Dynkin diagram. Their equations of motion are

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi_a = \frac{2m^2}{\beta}\left(e^{2\beta(\phi_{a+1}-\phi_a)} - e^{2\beta(\phi_a-\phi_{a-1})}\right). \tag{1}$$

Here and below, the index which enumerates nodes of the affine Dynkin diagram takes values in $\mathbb{Z}/N$. In particular, $\phi_{N+a} \equiv \phi_a$. Equations of motion (1) are generated by the following Hamiltonian and Poisson structure:

$$H = \sum_{a=1}^{N}\int dx\left(\frac{1}{2}\pi_a^2 + \frac{1}{2}(\partial_x\phi_a)^2 + \frac{m^2}{\beta^2}\,e^{2\beta(\phi_{a+1}-\phi_a)}\right), \tag{2}$$

$$\{\pi_a(x), \phi_b(y)\} = \delta_{ab}\,\delta(x-y). \tag{3}$$

The model under consideration is integrable. It admits a zero curvature representation with the following U–V pair [1,2]:

$$U(\lambda) = \sum_{a=1}^{N}\beta\,\pi_a\,e_{aa} + m\sum_{a=1}^{N}e^{\beta(\phi_{a+1}-\phi_a)}\left(\lambda^{\delta_{a,N}}e_{a,a+1} + \lambda^{-\delta_{a,N}}e_{a+1,a}\right), \tag{4}$$

$$V(\lambda) = \sum_{a=1}^{N}\beta\,\partial_x\phi_a\,e_{aa} + m\sum_{a=1}^{N}e^{\beta(\phi_{a+1}-\phi_a)}\left(\lambda^{\delta_{a,N}}e_{a,a+1} - \lambda^{-\delta_{a,N}}e_{a+1,a}\right), \tag{5}$$

where the e_{ab} are the basis matrices such that $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$.

The matrix U satisfies the following relation (the so-called fundamental Poisson bracket, see [3]):

$$\big\{U_1(\lambda), U_2(\mu)\big\} = \left[r\left(\frac{\lambda}{\mu}\right), U_1(\lambda) + U_2(\mu)\right], \tag{6}$$

where $r(\lambda)$ is the classical trigonometric r-matrix for the algebra A_{N-1}, see [3–5]. Here and below, lower indices denote the tensor component, e.g., $U_1 = U \otimes \mathbb{I}$.

1.2. Quantum lattice model. It is known that direct quantization of a continuous interacting field theory has problems with ultraviolet divergences. A possible roundabout is to consider a discrete regularization of the model by putting it on a one-dimensional lattice with step Δ. For the lattice model, quantum canonical variables that sit at different sites commute, and those that sit at the same site satisfy the following relations:

$$[\pi_a, \phi_b] = -i\,\hbar\,\delta_{ab}. \tag{7}$$

The classical continuous limit of these relations recovers the Poisson structure (3) if one assumes that

$$\pi_a^{(n)} = \Delta\,\pi_a(x), \quad \phi_a^{(n)} = \phi_a(x), \quad x = n\Delta, \tag{8}$$

where n is the lattice site number (it will be omitted in the subsequent formulas).

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Given an integrable classical continuous model, its quantum lattice analog is integrable as well if there exists a quantum L-operator (see, e.g., [6]) such that:

(i) its classical continuous limit recovers the corresponding matrix U:

$$L(\lambda)\Big|_{\hbar=0} = \mathbb{I} + \Delta\, U(\lambda) + o(\Delta)\,; \tag{9}$$

(ii) it satisfies the following quadratic commutation relation which is a lattice analog of the fundamental Poisson bracket (6):

$$R\left(\frac{\lambda}{\mu}\right)\, L_1(\lambda)\, L_2(\mu) = L_2(\mu)\, L_1(\lambda)\; R\left(\frac{\lambda}{\mu}\right). \tag{10}$$

The quantum R-matrix must satisfy the Yang–Baxter relation,

$$R_{12}\left(\frac{\lambda}{\mu}\right) R_{13}(\lambda)\, R_{23}(\mu) = R_{23}(\mu)\, R_{13}(\lambda)\, R_{12}\left(\frac{\lambda}{\mu}\right), \tag{11}$$

and its classical limit must recover the classical r-matrix.

For the $A^{(1)}_{N-1}$ Toda model, the quantum R-matrix has the following form [4, 5]:

$$R(\lambda) = \sum_{a,b=1}^{N} (\lambda q^{\delta_{ab}} - q^{-\delta_{ab}})\, e_{aa} \otimes e_{bb} + (q - q^{-1}) \sum_{a\neq b}^{N} \lambda^{\theta_{ab}}\, e_{ab} \otimes e_{ba}\,, \tag{12}$$

where $q = e^{i\beta^2\hbar}$, $\theta_{ab} = 0$ for $a < b$, and $\theta_{ab} = 1$ for $a > b$.

2. Lattice quantum L-operator

2.1. First order. We use the following notation:

$$\Pi = \mathrm{diag}(\pi_1, \ldots, \pi_N)\,, \qquad \Phi = \mathrm{diag}(\phi_1, \ldots, \phi_N)\,,$$

$$\widehat{e}_a = \lambda^{\delta_{a,N}} e_{a,a+1}\,, \qquad \widehat{f}_a = \lambda^{-\delta_{a,N}} e_{a+1,a}\,,$$

$$\widehat{E} = \sum_{a=1}^{N} \widehat{e}_a\,, \qquad \widehat{F} = \sum_{a=1}^{N} \widehat{f}_a\,.$$

In the seminal paper [5], M. Jimbo has found an approximate quantum L-operator for the $A^{(1)}_{N-1}$ Toda model. Namely, he showed that the following L-operator:

$$L^J(\lambda) = e^{\frac{\beta}{2}\Pi}\left(\mathbb{I} + \varepsilon\big(e^{-\beta \mathrm{ad}_\Phi}\widehat{E} + e^{\beta \mathrm{ad}_\Phi}\widehat{F}\big)\right) e^{\frac{\beta}{2}\Pi} = \sum_{a=1}^{N} e_{aa} e^{\beta\pi_a} + \varepsilon\, e^{\frac{\beta}{2}\Pi}\left(\sum_{a=1}^{N} e^{\beta(\phi_{a+1}-\phi_a)}\big(\widehat{e}_a + \widehat{f}_a\big)\right) e^{\frac{\beta}{2}\Pi} \tag{13}$$

satisfies the RLL-relations (10) in zero and first orders in ε.

It is easy to see that (13) satisfies condition (9) if we set

$$\varepsilon = m\Delta \tag{14}$$

and take into account the “renormalization” of momenta (8) in the continuous limit.

Note that, though the L-operator (13) is approximate, the corresponding R-matrix (12) contains no small parameter ε and is an exact solution to (11). In order to treat the quantum Toda model by means of the quantum inverse scattering method (see [6]), one needs an exact quantum L-operator that solves relation (10) in all orders in ε. In the present paper, we consider the second and third order corrections to the L-operator (13).

2.2. Second order. Consider an L-operator $L(\lambda,\varepsilon)$ that admits a series expansion in parameter ε,

$$L(\lambda,\varepsilon) = \sum_{n\geq 0} \varepsilon^n L^{(n)}(\lambda)\,. \tag{15}$$

Expanding relation (10) in ε, we obtain an infinite set of relations for $L^{(n)}(\lambda)$. Their closed forms corresponding to orders ε^n, $n = 0, 1, 2, 3$, are as follows:

$$R\Big(\frac{\lambda}{\mu}\Big)\, L_1^{(0)}(\lambda)\, L_2^{(0)}(\mu) = L_2^{(0)}(\mu)\, L_1^{(0)}(\lambda)\, R\Big(\frac{\lambda}{\mu}\Big), \tag{16}$$

$$R\Big(\frac{\lambda}{\mu}\Big)\,\big(L_1^{(1)}(\lambda)\, L_2^{(0)}(\mu) + L_1^{(0)}(\lambda)\, L_2^{(1)}(\mu)\big) = \big(L_2^{(1)}(\mu)\, L_1^{(0)}(\lambda) + L_2^{(0)}(\mu)\, L_1^{(1)}(\lambda)\big)\, R\Big(\frac{\lambda}{\mu}\Big), \tag{17}$$

$$\begin{aligned} R\Big(\frac{\lambda}{\mu}\Big)\,&\big(L_1^{(2)}(\lambda)\, L_2^{(0)}(\mu) + L_1^{(0)}(\lambda)\, L_2^{(2)}(\mu) + L_1^{(1)}(\lambda)\, L_2^{(1)}(\mu)\big) \\ &= \big(L_2^{(2)}(\mu)\, L_1^{(0)}(\lambda) + L_2^{(0)}(\mu)\, L_1^{(2)}(\lambda) + L_2^{(1)}(\mu)\, L_1^{(1)}(\lambda)\big)\, R\Big(\frac{\lambda}{\mu}\Big), \end{aligned} \tag{18}$$

and

$$\begin{aligned} R\Big(\frac{\lambda}{\mu}\Big)\,&\big(L_1^{(3)}(\lambda)\, L_2^{(0)}(\mu) + L_1^{(0)}(\lambda)\, L_2^{(3)}(\mu) + L_1^{(2)}(\lambda)\, L_2^{(1)}(\mu) + L_1^{(1)}(\lambda)\, L_2^{(2)}(\mu)\big) \\ &= \big(L_2^{(3)}(\mu)\, L_1^{(0)}(\lambda) + L_2^{(0)}(\mu)\, L_1^{(3)}(\lambda) + L_2^{(2)}(\mu)\, L_1^{(1)}(\lambda) + L_2^{(1)}(\mu)\, L_1^{(2)}(\lambda)\big)\, R\Big(\frac{\lambda}{\mu}\Big). \end{aligned} \tag{19}$$

We take

$$L^{(0)}(\lambda) = e^{\beta\Pi} \quad \text{and} \quad L^{(1)}(\lambda) = e^{\frac{\beta}{2}\Pi}\,\big(\rho_+\, e^{-\beta \mathrm{ad}_\Phi} \widehat{E} + \rho_-\, e^{\beta \mathrm{ad}_\Phi} \widehat{F}\big)\, e^{\frac{\beta}{2}\Pi}\,. \tag{20}$$

Notice that we have slightly generalized the first-order L-operator (13) by introducing arbitrary coefficients ρ_+ and ρ_-. In order to comply with the classical limit condition (9), we have to assume that $\rho_+, \rho_- \to 1$ as $\hbar \to 0$.

The problem which we want to solve is the following. First, given $L^{(0)}(\lambda)$ and $L^{(1)}(\lambda)$ as in (20), find the most general solution $L^{(2)}(\lambda)$ to Eqs. (18). Then we check whether Eq. (19) has a solution $L^{(3)}(\lambda)$ for some suitable $L^{(2)}(\lambda)$.

The main result of the present paper is the following statement.

Proposition 1. *Let $R(\lambda)$ be given by* (12), *and let $L^{(0)}(\lambda)$ and $L^{(1)}(\lambda)$ be given by* (20). *Then*

(i) *the general solution to Eq.* (18) *is given by*

$$\widetilde{L}^{(2)}(\lambda) = L^{(2)}(\lambda) + \widetilde{L}^{(1)}(\lambda)\,. \tag{21}$$

Here $\widetilde{L}^{(1)}(\lambda)$ is an arbitrary solution to Eq. (17), *and $L^{(2)}(\lambda)$ has the form*

$$L^{(2)}(\lambda) = e^{\frac{\beta}{2}\Pi}\Big(\gamma_1\, e^{-\beta \mathrm{ad}_\Phi} \widehat{E}^2 + \gamma_2\, e^{\beta \mathrm{ad}_\Phi} \widehat{F}^2 + \gamma_3\,(e^{-\beta \mathrm{ad}_\Phi} \widehat{E})(e^{\beta \mathrm{ad}_\Phi} \widehat{F}) + \gamma_4\,(e^{\beta \mathrm{ad}_\Phi} \widehat{F})(e^{-\beta \mathrm{ad}_\Phi} \widehat{E})\Big) e^{\frac{\beta}{2}\Pi}\,, \tag{22}$$

where the coefficients γ_i must satisfy the following conditions:

$$\textit{for } N = 2: \quad \gamma_3 + \gamma_4 = \rho_+ \rho_-\,; \tag{23}$$

$$\textit{for } N \geq 3: \quad \gamma_1 = \frac{q\,\rho_+^2}{1+q}, \quad \gamma_2 = \frac{\rho_-^2}{1+q}, \quad \textit{and} \quad \gamma_3 + \gamma_4 = \rho_+ \rho_-\,; \tag{24}$$

(ii) *for any choice of $\widetilde{L}^{(1)}(\lambda)$ in (21), Eq.* (19) *has no solution for $L^{(3)}(\lambda)$.*

A proof is given in Appendix A.

Formula (21) reflects the fact that the general solution to an inhomogeneous equation is the sum of its particular solution and the general solution of the corresponding homogeneous equation. Let us note that $\widetilde{L}^{(1)}(\lambda)$ does not necessarily satisfy condition (9).

The explicit expression for (22) involving the basis matrices is as follows:

$$\begin{aligned} L^{(2)}(\lambda) = &\sum_{a=1}^{N} e^{\frac{\beta}{2}\pi_a}\big(\gamma_3\, e^{2\beta(\phi_{a+1}-\phi_a)} + \gamma_4\, e^{2\beta(\phi_a-\phi_{a-1})}\big) e^{\frac{\beta}{2}\pi_a}\, e_{aa} \\ &+ \gamma_1 \sum_{a=1}^{N} e^{\frac{\beta}{2}\pi_{a-1}}\, e^{\beta(\phi_{a+1}-\phi_{a-1})}\, e^{\frac{\beta}{2}\pi_{a+1}}\, e_{a-1,a+1}\, \lambda^{\delta_{a,1}+\delta_{a,N}} \\ &+ \gamma_2 \sum_{a=1}^{N} e^{\frac{\beta}{2}\pi_{a+1}}\, e^{\beta(\phi_{a+1}-\phi_{a-1})}\, e^{\frac{\beta}{2}\pi_{a-1}}\, e_{a+1,a-1}\, \lambda^{\delta_{a,1}+\delta_{a,N}}\,. \end{aligned} \tag{25}$$

For $N = 2$, Eq. (25) contains only diagonal terms. In this case, choosing $\rho_+ = \rho_- = 1$, $\gamma_1 = \gamma_2 = 0$, and $\gamma_3 = 1$ (or $\gamma_3 = 0$), we obtain an exact L-operator,

$$L(\lambda) = \begin{pmatrix} e^{\frac{\beta}{2}\pi_1}\big(1 + \varepsilon^2 e^{2\widetilde{\beta}(\phi_2-\phi_1)}\big)e^{\frac{\beta}{2}\pi_1} & \varepsilon e^{\frac{\beta}{2}\pi_1}\big(e^{\beta(\phi_2-\phi_1)} + \lambda^{-1}e^{\beta(\phi_1-\phi_2)}\big)e^{\frac{\beta}{2}\pi_2} \\ \varepsilon e^{\frac{\beta}{2}\pi_2}\big(e^{\beta(\phi_2-\phi_1)} + \lambda e^{\beta(\phi_1-\phi_2)}\big)e^{\frac{\beta}{2}\pi_1} & e^{\frac{\beta}{2}\pi_2}\big(1 + \varepsilon^2 e^{2\widetilde{\beta}(\phi_1-\phi_2)}\big)e^{\frac{\beta}{2}\pi_2} \end{pmatrix}, \tag{26}$$

where $\widetilde{\beta} = (\gamma_3 - \gamma_4)\beta$. The L-operator (26) satisfies relation (10) in all orders in ε. Upon the reduction $\phi_2 = -\phi_1$, $\pi_2 = -\pi_1$, Eq. (26) yields the well-known exact L-operator for the sinh-Gordon model [6,7].

3. Reduction to nonaffine case

The (1+1)-dimensional Toda chain associated with the simple Lie algebra A_{N-1} describes the relativistic dynamics of N scalar fields whose equations of motion are given by the same Eq. (1), where no periodicity in index a is assumed. In this case, one can formally set $\beta\phi_0 = -\beta\phi_{N+1} = +\infty$ in (1) and (2). The same procedure applied to (4)–(5) yields a U–V pair without a spectral parameter.

In order to keep the spectral parameter in the U–V pair, the following procedure was suggested in [8] (in the case corresponding to A_1; a generalization was considered in [9]). Take $\xi > 0$ and shift (the zero modes of) the fields, mass, and spectral parameter in (4)–(5) as follows:

$$\phi_a \to \phi_a + a\,\xi/\beta\,, \quad m \to e^{-\xi}\,m\,, \quad \lambda \to e^{\xi N}\lambda\,. \tag{27}$$

Then the limit as $\xi \to +\infty$ yields the following U–V pair:

$$U(\lambda) = \beta\Pi + m\big(e^{-\beta\mathrm{ad}_\Phi}\widehat{E} + e^{\beta\mathrm{ad}_\Phi}F\big) \quad \text{and} \quad V(\lambda) = \beta\partial_x\Phi + m\big(e^{-\beta\mathrm{ad}_\Phi}\widehat{E} - e^{\beta\mathrm{ad}_\Phi}F\big), \tag{28}$$

where

$$\widehat{E} = \sum_{a=1}^{N}\widehat{e}_a = \sum_{a=1}^{N}\lambda^{\delta_{a,N}}e_{a,a+1} \quad \text{and} \quad F = \sum_{a=1}^{N-1}\widehat{f}_a = \sum_{a=1}^{N-1}e_{a+1,a}. \tag{29}$$

The U-matrix in (28) satisfies the same fundamental Poisson bracket (6) with the same classical r-matrix as in the affine case.

In the nonaffine case, we have the following counterpart of Proposition 1.

Proposition 2. *Let* $R(\lambda)$ *be given by* (12). *Then:*

(i) *Equations* (16) *and* (17) *admit the following solutions:*

$$L^{(0)}(\lambda) = e^{\beta\Pi} \quad \textit{and} \quad L^{(1)}(\lambda) = e^{\frac{\beta}{2}\Pi}\left(\rho_+\,e^{-\beta\mathrm{ad}_\Phi}\widehat{E} + \rho_-\,e^{\beta\mathrm{ad}_\Phi}F\right)e^{\frac{\beta}{2}\Pi}\,; \tag{30}$$

(ii) *given* $L^{(0)}(\lambda)$ *and* $L^{(1)}(\lambda)$ *as in Eq.* (30), *the general solution to Eq.* (18) *has the form*

$$\widetilde{L}^{(2)}(\lambda) = L^{(2)}(\lambda) + \widetilde{L}^{(1)}(\lambda)\,. \tag{31}$$

Here $\widetilde{L}^{(1)}(\lambda)$ *is an arbitrary solution to Eq.* (17), *and* $L^{(2)}(\lambda)$ *has the form*

$$L^{(2)}(\lambda) = e^{\frac{\beta}{2}\Pi}\Big(\gamma_1\,e^{-\beta\mathrm{ad}_\Phi}\widehat{E}^2 + \gamma_2\,e^{\beta\mathrm{ad}_\Phi}F^2 + \gamma_3\,(e^{-\beta\mathrm{ad}_\Phi}\widehat{E})(e^{\beta\mathrm{ad}_\Phi}F) + \gamma_4\,(e^{\beta\mathrm{ad}_\Phi}F)(e^{-\beta\mathrm{ad}_\Phi}\widehat{E})\Big)e^{\frac{\beta}{2}\Pi}\,, \tag{32}$$

where the coefficients γ_i *must satisfy conditions* (23) *and* (24);

(iii) *for any choice of* $\widetilde{L}^{(1)}(\lambda)$ *in* (31), *Eq.* (19) *has no solution for* $L^{(3)}(\lambda)$.

A proof is given in the Appendix.

For $N = 2$, Eq. (32) contains only diagonal terms. Furthermore, $F^2 = 0$. In this case, choosing $\rho_+ = \rho_- = 1$, $\gamma_1 = 0$, and $\gamma_3 = 1$ (or $\gamma_3 = 0$), we obtain an exact L-operator,

$$L(\lambda) = \begin{pmatrix} e^{\frac{\beta}{2}\pi_1}\big(1 + \varepsilon^2 e^{2\widetilde{\beta}(\phi_2-\phi_1)}\big)e^{\frac{\beta}{2}\pi_1} & \varepsilon e^{\frac{\beta}{2}\pi_1}e^{\beta(\phi_2-\phi_1)}e^{\frac{\beta}{2}\pi_2} \\ \varepsilon e^{\frac{\beta}{2}\pi_2}\big(e^{\beta(\phi_2-\phi_1)} + \lambda e^{\beta(\phi_1-\phi_2)}\big)e^{\frac{\beta}{2}\pi_1} & e^{\frac{\beta}{2}\pi_2}\big(1 + \varepsilon^2 e^{2\widetilde{\beta}(\phi_1-\phi_2)}\big)e^{\frac{\beta}{2}\pi_2} \end{pmatrix}, \tag{33}$$

where $\widetilde{\beta} = (\gamma_3 - \gamma_4)\beta$. The L-operator (33) satisfies relation (10) in all orders in ε. Upon the reduction $\phi_2 = -\phi_1$, $\pi_2 = -\pi_1$, Eq. (33) yields the exact L-operator for the Liouville model [8].

Appendix A

A.1. *Proof of Proposition 1.* Second order. We use the following notation:

$$H_a = e_{aa} - e_{a+1,a+1}\,, \quad K_a = q^{\frac{1}{2}H_a}\,, \quad \text{and} \quad \alpha_a(X) = \mathrm{tr}(H_a X)\,,$$

where $a = 1, \ldots, N$ and $e_{N+1,N+1} \equiv e_{11}$. Then

$$K_a \widehat{e}_b = q^{\frac{1}{2}A_{ab}}\,\widehat{e}_b K_a \quad \text{and} \quad K_a \widehat{f}_b = q^{-\frac{1}{2}A_{ab}}\,\widehat{f}_b K_a\,, \tag{34}$$

where A is the Cartan matrix of the affine algebra $A^{(1)}_{N-1}$.

In the first step, following [5], we rewrite $L^{(1)}(\lambda)$ and $L^{(2)}(\lambda)$ by moving $e^{\frac{\beta}{2}\Pi}$ to the extreme right:

$$L^{(1)}(\lambda) = \sum_{a=1}^{N} e^{-\beta\alpha_a(\Phi)}\Big(\rho_+ e^{\frac{\beta}{2}\alpha_a(\Pi)} K_a \widehat{e}_a + \rho_- e^{-\frac{\beta}{2}\alpha_a(\Pi)} K_a \widehat{f}_a\Big) e^{\beta\Pi} \tag{35}$$

and

$$\begin{aligned} L^{(2)}(\lambda) = \sum_{a=1}^{N}\Big(&\gamma_1\, e^{-\beta(\alpha_a(\Phi)+\alpha_{a+1}(\Phi))} e^{\frac{\beta}{2}(\alpha_a(\Pi)+\alpha_{a+1}(\Pi))} K_a K_{a+1} \widehat{e}_a \widehat{e}_{a+1} \\ &+ \gamma_2\, e^{-\beta(\alpha_a(\Phi)+\alpha_{a+1}(\Phi))} e^{-\frac{\beta}{2}(\alpha_a(\Pi)+\alpha_{a+1}(\Pi))} K_a K_{a+1} \widehat{f}_{a+1} \widehat{f}_a + e^{-2\beta\alpha_a(\Phi)} K_a^2 \big(\gamma_3\, \widehat{e}_a \widehat{f}_a + \gamma_4\, \widehat{f}_a \widehat{e}_a\big)\Big) e^{\beta\Pi}. \end{aligned} \tag{36}$$

Next, we substitute (35)–(36) into (17)–(18) and move all the factors containing $e^{\beta\Pi}$ to the right using the relations

$$e^{\beta\Pi_1} e^{-\beta\alpha_a(\Phi)} = e^{-\beta\alpha_a(\Phi)} \big(K_a^2 \otimes \mathbb{I}\big)\, e^{\beta\Pi_1} \quad \text{and} \quad e^{\beta\Pi_2} e^{-\beta\alpha_a(\Phi)} = e^{-\beta\alpha_a(\Phi)} \big(\mathbb{I} \otimes K_a^2\big)\, e^{\beta\Pi_2}.$$

Finally, matching coefficients at functionally independent exponentials of quantum fields, we obtain a set of relations. Here one should take into account that $R(\lambda)$ commutes with $(e_{aa} \otimes \mathbb{I} + \mathbb{I} \otimes e_{aa})$; hence,

$$[R(\lambda), K_a \otimes K_a] = 0 \quad \text{and} \quad [R(\lambda), e^{\beta\Pi_1} e^{\beta\Pi_2}] = 0\,.$$

The relations which arise as matching conditions for coefficients in (17) at the fields $e^{-\beta\alpha_a(\Phi)} e^{\pm\frac{\beta}{2}\alpha_a(\Pi)}$ are as follows:

$$R\Big(\frac{\lambda}{\mu}\Big)\, \Delta(x_a) = \Delta'(x_a)\, R\Big(\frac{\lambda}{\mu}\Big), \quad a = 1, \ldots, N\,, \tag{37}$$

where $x_a = \widehat{e}_a, \widehat{f}_a$, respectively,

$$\Delta(x_a) = x_a \otimes K_a^{-1} + K_a \otimes x_a\,, \quad \text{and} \quad \Delta'(x_a) = x_a \otimes K_a + K_a^{-1} \otimes x_a\,. \tag{38}$$

Here and below, $x_N \otimes \mathbb{I}$ depends on λ while $\mathbb{I} \otimes x_N$ depends on μ.

In [5], Jimbo has shown that the solution to Eqs. (37) is unique up to an overall scalar factor and that this solution is given by the R-matrix (12).

Now, treating Eq. (18) similarly and matching coefficients at the fields

$$e^{-\beta(\alpha_a(\Phi)+\alpha_b(\Phi))} e^{\frac{\beta}{2}(\kappa_1\alpha_a(\Pi)+\kappa_2\alpha_b(\Pi))}, \quad \kappa_i = \pm\,,$$

we get the relations

$$R\Big(\frac{\lambda}{\mu}\Big)\, X_{ab}^{\kappa_1\kappa_2} = \big(X_{ab}^{\kappa_1\kappa_2}\big)'\, R\Big(\frac{\lambda}{\mu}\Big), \quad a, b = 1, \ldots, N\,, \tag{39}$$

where the prime at the right-hand side denotes permutation of tensor factors (similar to that in (38)). Obviously, $X_{ab}^{\kappa_1\kappa_2} = X_{ba}^{\kappa_2\kappa_1}$. We have the relations

$$X_{ab}^{++} = \widehat{e}_a K_b \otimes K_a^{-1} \widehat{e}_b + (1 - \delta_{ab})\, \widehat{e}_b K_a \otimes K_b^{-1} \widehat{e}_a \quad \text{for} \quad a - b \neq \pm 1 \bmod N, \tag{40}$$

$$X_{a,a+1}^{++} = \gamma_1 \big(\widehat{e}_a \widehat{e}_{a+1} \otimes K_a^{-1} K_{a+1}^{-1} + K_a K_{a+1} \otimes \widehat{e}_a \widehat{e}_{a+1}\big) + \rho_+^2 \big(\widehat{e}_a K_{a+1} \otimes K_a^{-1} \widehat{e}_{a+1} + \widehat{e}_{a+1} K_a \otimes K_{a+1}^{-1} \widehat{e}_a\big)\,, \tag{41}$$

$$X_{ab}^{--} = \widehat{f}_a K_b \otimes K_a^{-1} \widehat{f}_b + (1 - \delta_{ab})\, \widehat{f}_b K_a \otimes K_b^{-1} \widehat{f}_a \quad \text{for} \quad a - b \neq \pm 1 \bmod N, \tag{42}$$

$$X_{a,a+1}^{--} = \gamma_2 \big(\widehat{f}_{a+1} \widehat{f}_a \otimes K_a^{-1} K_{a+1}^{-1} + K_a K_{a+1} \otimes \widehat{f}_{a+1} \widehat{f}_a\big) + \rho_-^2 \big(\widehat{f}_a K_{a+1} \otimes K_a^{-1} \widehat{f}_{a+1} + \widehat{f}_{a+1} K_a \otimes K_{a+1}^{-1} \widehat{f}_a\big)\,, \tag{43}$$

and

$$X^{+-}_{ab} = \delta_{ab}\Big(\gamma_3\left(\widehat{e}_a\widehat{f}_a \otimes K_a^{-2} + K_a^2 \otimes \widehat{e}_a\widehat{f}_a\right) + \gamma_4\left(\widehat{f}_a\widehat{e}_a \otimes K_a^{-2} + K_a^2 \otimes \widehat{f}_a\widehat{e}_a\right)\Big)$$
$$+ \rho_+\rho_-\left(\widehat{e}_a K_b \otimes K_a^{-1}\widehat{f}_b + \widehat{f}_b K_a \otimes K_b^{-1}\widehat{e}_a\right). \tag{44}$$

Let us note that Eqs. (34) and (38) are relations for generators of the affine algebra $A^{(1)}_{N-1}$ which hold for any rank and representation. However, in the case of the fundamental representation, we have extra relations:

$$\text{for } N \geq 2: \quad \widehat{e}_a\widehat{f}_b = \widehat{f}_b\widehat{e}_a = 0 \quad \text{if } \ b \neq a\,;$$
$$\text{for } N \geq 3: \quad \widehat{e}_a\widehat{e}_b = \widehat{f}_b\widehat{f}_a = 0 \quad \text{if } \ b - a \neq 1 \bmod N\,.$$

Taking them into account, we note that

$$(1+q^2)X^{++}_{aa} = \Delta(\widehat{e}_a)\Delta(\widehat{e}_a)\,, \quad (1+q^2)X^{--}_{aa} = \Delta(\widehat{f}_a)\Delta(\widehat{f}_a)\,, \tag{45}$$

$$X^{++}_{ab} = \Delta(\widehat{e}_a)\Delta(\widehat{e}_b) \quad \text{and} \quad X^{--}_{ab} = \Delta(\widehat{f}_a)\Delta(\widehat{f}_b) \quad \text{for } \ a - b \neq 0, \pm 1 \bmod N\,, \tag{46}$$

$$X^{+-}_{ab} = \rho_+\rho_-\,\Delta(\widehat{e}_a)\Delta(\widehat{f}_b) \quad \text{for } \ a \neq b\,, \tag{47}$$

$$X^{++}_{a,a+1} - \gamma_1\,\Delta(\widehat{e}_a)\Delta(\widehat{e}_{a+1}) - q(\rho_+^2 - \gamma_1)\,\Delta(\widehat{e}_{a+1})\Delta(\widehat{e}_a) = \big((1-q)\rho_+^2 + (q - q^{-1})\gamma_1\big)\,\widehat{e}_{a+1}K_a \otimes K^{-1}_{a+1}\widehat{e}_a\,, \tag{48}$$

$$X^{--}_{a,a+1} - \gamma_2\,\Delta(\widehat{f}_{a+1})\Delta(\widehat{f}_a) - q^{-1}(\rho_-^2 - \gamma_2)\,\Delta(\widehat{f}_a)\Delta(\widehat{f}_{a+1})$$
$$= \big((1-q^{-1})\rho_-^2 + (q^{-1} - q)\gamma_2\big)\,\widehat{f}_a K_{a+1} \otimes K_a^{-1}\widehat{f}_{a+1}\,, \tag{49}$$

and

$$X^{+-}_{aa} - \gamma_3\,\Delta(\widehat{e}_a)\Delta(\widehat{f}_a) - \gamma_4\,\Delta(\widehat{f}_a)\Delta(\widehat{e}_a) = (1 - \gamma_3 - \gamma_4)\left(\widehat{e}_a K_a \otimes K_a^{-1}\widehat{f}_a + \widehat{f}_a K_a \otimes K_a^{-1}\widehat{e}_a\right). \tag{50}$$

Thus, the condition that the considered $L^{(2)}$ is a solution to (18) is equivalent to the requirement that relation (39) holds for the right-hand side of (45)–(50). Equations (37)–(38) imply that the right-hand side of (45)–(47) satisfies (39). Furthermore, it is straightforward to check that (39) does not hold for the right-hand side of (48)–(50). This implies that scalar factors on the right-hand side of these equations must vanish. In this way, we obtain the values of γ_i given in Proposition 1.

A.2. *Proof of Proposition 1.* Third order.

Lemma 1. *Let $R(\lambda)$ be given by* (12) *and let $L^{(0)}(\lambda)$ be as in* (20)*. Let*

$$\widetilde{L}^{(1)}(\lambda) = \sum_{a,b=1}^{N} \widetilde{L}^{(1)}_{ab}(\lambda)\, e_{ab} \tag{51}$$

be an arbitrary solution to Eq. (17)*. Then the operator-valued coefficients $\widetilde{L}^{(1)}_{ab}(\lambda)$ vanish unless $a = b$ or $a - b = \pm 1 \bmod N$.*

Proof. Consider the matrix entry $e_{cb} \otimes e_{ac}$ of Eq. (17). Choose a, b, c such that $a \neq b$, $a \neq c$, and $b \neq c$. Then, since $L^{(0)}(\lambda)$ is a diagonal matrix, the computation of the matrix element in question involves only the nondiagonal part of the R-matrix (12). It is straightforward to check that as a result, we obtain the following equation:

$$\left(\frac{\lambda}{\mu}\right)^{\theta_{ca}} \widetilde{L}^{(1)}_{ab}(\lambda) = \left(\frac{\lambda}{\mu}\right)^{\theta_{cb}} \widetilde{L}^{(1)}_{ab}(\mu)\,. \tag{52}$$

Now, if $b - a \neq 0, \pm 1 \bmod N$, then (52) for $c = a - 1 \bmod N$ and $c = a + 1 \bmod N$ yields two equations which are inconsistent unless $\widetilde{L}^{(1)}_{ab}(\lambda) = 0$. This completes the proof of Lemma 1. □

In order to prove part (ii) of Proposition 1, we write

$$\widetilde{L}^{(2)}(\lambda) = \sum_{a,b=1}^{N} \widetilde{L}^{(2)}_{ab}(\lambda)\, e_{ab} \quad \text{and} \quad L^{(3)}(\lambda) = \sum_{a,b=1}^{N} L^{(3)}_{ab}(\lambda)\, e_{ab} \tag{53}$$

for the general solution of (18) and for the sought solution of (19).

Consider the matrix entry $e_{a,a+1} \otimes e_{a-1,a+1}$ of Eq. (19) in the case where $N \geq 3$. It is straightforward to check that the resulting equation reads as follows:

$$\left(\frac{\lambda}{\mu} - 1\right) L^{(1)}_{a,a+1}(\lambda) \widetilde{L}^{(2)}_{a-1,a+1}(\mu) + (q - q^{-1}) \frac{\lambda}{\mu} \widetilde{L}^{(2)}_{a-1,a+1}(\lambda) L^{(1)}_{a,a+1}(\mu) = \left(q\frac{\lambda}{\mu} - q^{-1}\right) \widetilde{L}^{(2)}_{a-1,a+1}(\mu) L^{(1)}_{a,a+1}(\lambda) . \quad (54)$$

Note that this equation, although it is resulting from the third order in the ε-expansion, does not involve matrix entries of $L^{(3)}$. The reason is that in (19), $L^{(3)}$ is coupled to $L^{(0)}$ for which the matrix entries $(a, a+1)$ and $(a-1, a+1)$ vanish.

Now, by Lemma 1, we can replace $\widetilde{L}^{(2)}$ in (54) by the particular $L^{(2)}$ given by (22) since they must have coinciding matrix entries $(a-1, a+1)$. Finally, it is easy to check that (54) does not hold for matrix entries of $L^{(1)}$ and $L^{(2)}$ (cf. (13) and (25)). Therefore, for any possible choice of $\widetilde{L}^{(2)}$, Eq. (19) has no solution $L^{(3)}$.

A.3. *Proof of Proposition 2.* Part (i). We can obtain $L^{(1)}$ in (30) from $L^{(1)}$ in (20) by setting $\widehat{f}_N = 0$. Since relations (37) are linear in x_a, they are consistent with such a reduction. Hence, it follows that $L^{(1)}$ given by (30) is a solution to (18).

Part (ii). Similarly, setting $\widehat{f}_N = 0$ in (22), we obtain (32). A direct inspection of (40)–(44) shows that the X^{++}_{ab} are not affected by the reduction, while X^{--}_{ab} and X^{+-}_{ab} vanish if $a = N$ or $b = N$ and do not change if $a, b \neq N$. Therefore, relations (39) remain valid, which, in turn, implies that (19) holds.

Part (iii). It suffices to repeat the reasoning given in Sec. A.2 and notice that the matrix entries $L^{(1)}_{a,a+1}$ and $L^{(2)}_{a-1,a+1}$ are not affected by the reduction. $\square$

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